REMARKS ON THE STRONG LIFTING PROPERTY FOR PRODUCTS

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ABSTRACT

The class of compact measure spaces which possesses the attribute of having products with the strong lifting property is much larger than that of the metric spaces. This class includes every homogeneous space equipped with a quasi-invariant measure. This result, in conjunction with Losert's example and Kupka's arguments, yields invariant measures on transformation groups, which fail to have a lifting commuting with left translations. In addition, the previously mentioned class contains every product measure on an arbitrary product of metrizable spaces.

1. Introduction and review of terminology

This paper reviews concepts and extends some results of [4].

J. Kupka posed the question as to what extent the product of two compact measure spaces with the strong lifting property admits a strong lifting [8, 3.3 Question]. In the bounded case, the answer to this question is always positive, if one, at least, space is metrizable [5, th. 4, p. 115]. Roughly speaking, the measures on metric spaces "preserve the strong lifting property on products". It is therefore obvious that the determination of all compact measure spaces which preserve the strong lifting property on products, answers the above question. Thus, this "product strong lifting property" is worth exploring.

Theorem 2.6 asserts that a given quasi-invariant measure on a compact homogeneous space (and therefore the Haar measure on any compact group) has the previously cited property. Theorem 2.7 presents another class of measure spaces with this property. A special case of such spaces is any product of metrizable spaces.

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1.1. A compact measure space is a pair (T,μ), where T is a compact topological space and μ a (positive, Radon) measure on T. If there exists a strong (resp. almost strong) lifting for (T,μ), then (T,μ) (or just μ) is said to have the strong (resp. almost strong) lifting property (in brief: SLP, resp. ASLP (cf. [5]), the term compact inferred and the term measure space used instead.

1.2. Let (T,μ), (X,λ) be measure spaces and $p: X\to T$ a continuous surjection such that $p(\lambda)=\mu$. Then a family $\{\lambda_t, t\in T\}$ of measures on X will be said to constitute a strict p-disintegration of λ ([5], [8]), if for every function $f\geqq 0$, on X, measurable the following conditions are satisfied:

(1) f is λ_t-measurable a.e. (μ).

(2) The (μ-almost everywhere defined) function $t\to\lambda_t(f)$ is μ-measurable and for every subset A of T μ-measurable, we have

$$\int_{p^{-1}(A)} f d\lambda = \int_A \lambda_t(f)d\mu(t).$$

(3) $\operatorname{Supp}(\lambda_t)\subset p^{-1}\{t\}$ a.e. (μ).

1.3. Let (H,X) be a (right) compact transformation group, where X is a compact space and H a compact topological group which acts on X by (the action): $H\times X\to X$: $(\xi,x)\to x\xi$. Let X/H denote the space of (right) cosets of X and π: $X\to X/H$: $x\to\dot{x}$ the canonical mapping. Let β be a (left) Haar measure on H and λ a measure on X/H. Then (cf. [2]) there is a unique measure λ^* on X (the so-called Haar lift of λ) with the property

$$\lambda^*(f)=\int_{X/H} d\lambda(\dot{x})\int_H f(x\xi)d\beta(\xi), \qquad f\in C(X).$$

In particular, if $X=G$ is a compact group and H a closed subgroup of G, one may determine a quasi-invariant measure λ on (the homogeneous space) G/H ([2], [8]). Note that a transformation group is called free, if $\xi\in H$, $\xi\neq e$ then $x\xi\neq x$, $x\in X$, where e is the identity of G.

In Section 2 the definition of product-strong lifting property is given and some results concerning this property are presented.

2. The product-strong lifting property

The following concept is introduced:

2.1. Definition. Let us say a measure space (X,λ) (or just λ) has the product-strong lifting property (in brief: $\otimes$SLP), if for every measure space

(T, μ) with the strong lifting property, the product measure $\lambda \otimes \mu$ has the strong lifting property.

2.2. LEMMA. *Let (H, X) be a compact transformation group. Then, for an arbitrary positive measure λ on X/H with full support, the following assertions are valid:*

(1) *$(X/H, \lambda)$ has $\otimes$SLP, if (X, λ^*) has $\otimes$SLP.*

(2) *If in particular H acts freely on X, then $\otimes$SLP for λ is equivalent to $\otimes$SLP for λ^*.*

PROOF. (1) The proof of (1) essentially uses arguments due to Kupka [8]. For $x \in X$, the mapping $p_x : H \to X : \xi \to p_x(\xi) := x\xi$ is continuous and for $u = \pi(x) = \dot{x} \in X/H$, the image $\beta_u := p_x(\beta)$ of the Haar measure β on H under p_x depends only from the class of x.

Since X is compact, $\lambda = \pi(\lambda^*)$. Furthermore, the family $\{\beta_u, u \in X/H\}$ is a strict π-disintegration of λ^*, which satisfies the special condition [2]

$$\text{“}u \to \beta_u(f) \text{ is continuous on } X/H, \text{ for } f \in C(X)\text{”}.$$

It will be shown that $(X/H, \lambda)$ has $\otimes$SLP. Consider a measure space (T, μ) with the strong lifting property. Then taking the continuous mapping $p: T \times X \to T \times (X/H)$: $(t, x) \to (t, \pi(x))$, one can immediately verify the following:

1. $p(\mu \otimes \lambda^*) = \mu \otimes \lambda$,
2. The family $\{\delta_t \otimes \beta_u, \ (t, u) \in T \times (X/H)\}$ (where δ_t denotes the Dirac measure on T, at t) is a strict p-disintegration of $\mu \otimes \lambda^*$.

By assumption, there is a strong lifting for $(T \times X, \mu \times \lambda^*)$. Consequently, the desired conclusion results from [8, projection theorem].

(2) Using the previous notations, it will be verified that $\mu \otimes \lambda^*$ has SLP, provided that $\mu \otimes \lambda$ has SLP. It is confirmed that H acts freely on $T \times X$ by the action $H \times T \times X \to T \times X$: $(\xi, t, x) \to (t, x\xi)$ and that this action satisfies

$$T \times (X/H) \simeq (T \times X)/H \quad \text{and} \quad (\mu \otimes \lambda)^* = \mu \otimes \lambda^*.$$

Since $(H, T \times X)$ is free, by the arguments of [6], it is clear that there will exist a (strong) lifting for $\mu \otimes \lambda^*$, commuting with this action.

2.3. REMARK. K. Bichteler [1] has noticed the fact that the set of Radon measures λ on a (locally) compact space X such that $(X, |\lambda|)$ has ASLP is a band of the space $M(X)$ of all measures on X. The problem as to whether or not every measure space (with full support) has $\otimes$SLP is open. However, by elementary arguments, it can also be shown that the set of all measures λ on X such that $|\lambda|$ "preserves ASLP on products" is a band of $M(X)$.

The following obvious proposition is of independent interest.

2.4. PROPOSITION. *Every compact group equipped with the Haar measure has* $\otimes$SLP.

2.5. DISCUSSION [concerning the examples of [8]]. Let us assume that B is a compact space which supports a measure λ, H any compact group and β the Haar measure on H. Then, according to the proof of Lemma 2.3, the couple $(H, X_H = B \times H)$ becomes a transformation group with respect to the operation

$$H \times B \times H \to B \times H\colon (\eta, (b, \xi)) \to (b, \xi\eta).$$

This operation satisfies: $X_H \sim B$ and $\lambda \otimes \beta = \lambda^*$. This model yields many examples of measures on transformation groups without SLP. For instance, if we take a space (B, λ) without SLP ([8], [9]), then for the corresponding (X_H, λ^*) there is no (strong) lifting, commuting with left translations.

2.6. THEOREM. *The quasi-invariant measure on a compact homogeneous space has* $\otimes$SLP.

PROOF. Let λ be the quasi-invariant measure on an homogeneous space G/H. Then (cf. [2]) the Haar lift λ^* must be equivalent to the Haar measure on G. Since, from 2.4, this Haar measure has $\otimes$SLP, the conclusion is obvious, from Lemma 2.3.

The next theorem and lemma generalize the fact that on any product of metric spaces the product measure has always the strong lifting property (cf. [5, Examples 1, p. 119]).

2.7. THEOREM. *Every continuous image of a product measure on an arbitrary product of metric spaces has* $\otimes$SLP, *if it has* SLP.

2.8. LEMMA. *Let* $X = S \times (\Pi_{i\in J} U_i)$, *where* $(U_i)_{i\in J}$ *is a family of compact metrizable spaces and* $S = U_{i_0}$ *a compact space. Let* λ *be a positive measure on* $\Pi_{i\in J} U_i$ *with* $\operatorname{Supp}\lambda = \Pi_{i\in J} U_i$ *and* ν *a positive measure on* S *with* $\operatorname{Supp}\nu = S$. *Suppose that*:

($\#$) *If* $I \subset J \cup \{i_0\}$, $i \in J \cup \{i_0\} - I$, $B \subset \Pi_{i\in J\cup\{i_0\}} U_i$ *is* $p_I(\nu\otimes\lambda)$*-measurable and* $V \subset U_i$ *is open, then* $p_{I\cup(i)}(\nu\otimes\lambda)(B\times V) = 0$ *implies* $p_I(\nu\otimes\lambda)(B)p_{\{i\}}(V) = 0$.

Then $(X, \nu\otimes\lambda)$ *has* SLP.

PROOF. The measure space $(S \times \Pi_{i\in J} U_i, \nu\otimes\lambda)$ is of the form $(X, \mu) := (\Pi_{k\in K} X_k, \nu\otimes\lambda)$. We propose to show that (X, μ) admits a strong lifting. Reproducing, step by step, the proof of [5, th. 5, p. 118], without any innovation, we establish the fact that this space has SLP. Summary steps for the proof can be sketched as follows:

1. We verify that μ satisfies the condition 5.2 of [5, th. 5, p. 118].

2. Using the notations and the content of [5, pp. 116–118] we see that the set Γ of all pairs (I, r^I), where r^I is a strong lifting for $(p_I(X), p_I(\mu))$, $I \subset K$, is inductive with respect to the order.

3. We check that a maximal element of Γ is a strong lifting for (X, μ).

2.9. Corollary. *A product measure on an arbitrary product of metrizable spaces has* $\otimes$SLP.

Proof. Let $(U = \Pi_{i \in J} U_i, \lambda = \bigotimes_{i \in J} \lambda_i)$ be a product of measure spaces, where U_i is metrizable and λ_i a probability measure on U_i, such that supp $\lambda_i = U_i$, for every $i \in J$. Let us assume that (S, ν) is any measure space with the strong lifting property. Then, since λ is a product measure, we easily verify that the space $(S \times U, \nu \otimes \lambda)$ satisfies the condition $(\#)$ of Lemma 2.8, therefore we have the conclusion.

2.10. Remark. The property $\otimes$SLP is not an exclusive privilege of the Haar measure on a group. In fact, on $\{0,1\}^\alpha$, where α is an arbitrary set, one can construct a measure λ which satisfies the conditions of Lemma 2.8 and which is not, generally, equivalent to a product measure.[†]

For each $p \in [0,1]$ let λ_p be the measure on $\{0,1\}$ defined by $\lambda_p(\{0\}) = p$ and $\lambda_p(\{1\}) = 1 - p$. For each $i \in \alpha$ let $\lambda_p^i = \lambda_p$. For every $\beta \subset \alpha$ define the measure λ_p^β on $\{0,1\}^\beta$ by $\lambda_p^\beta := \bigotimes_{i \in \beta} \lambda_p^i$.

Taking now a probability measure m on $[0,1]$ such that $m(\{0\}) = m(\{1\}) = 0$, define the measure λ on $\{0,1\}^\alpha$ by

$$\lambda := \int_{[0,1]} \lambda_p^\alpha dm(p).$$

Let now (S, ν) be any measure space with the strong lifting property. Then we immediately verify that the following is valid:

$$(\cdot) \qquad \nu \otimes p_\beta(\lambda) := \int_{[0,1]} \lambda_p^\beta dm(p), \qquad \text{for } \beta \subset \alpha.$$

Using $(\cdot)$ we can easily check that the space $(S \times \{0,1\}^\alpha, \nu \otimes \lambda)$ satisfies $(\#)$ of Lemma 2.8 and thus $(\{0,1\}^\alpha, \lambda)$ has $\otimes$SLP.

[†] Note the well-known example of [5].

PROOF OF THEOREM 2.7. Let (T, μ) be a measure space which satisfies the conditions:

(1) (T, μ) has SLP.

(2) There exist a family $\{(U_i, \lambda_i), i \in J\}$, where U_i is metrizable, λ_i a probability on U_i and a continuous map p from $\Pi_{i \in J} U_i$ onto T such that $p(\bigotimes_{i \in J} \lambda_i) = \mu$.

Now let (B, κ) be any space with SLP, (S, ν) the hyperstonian space associated with (B, κ) and $w: S \to B$ the canonical map. The following mapping is defined:

$$q: S \times U \to B \times T: (s, u) \to (w(s), p(u)).$$

Then $\kappa \otimes \mu = q(\nu \otimes \lambda)$ is easily verified. Moreover the following are valid:

(I) The measure $\nu \otimes \lambda$ on $S \times U$ is completion regular. In fact, since ν is completion regular, this is clear by the arguments of [3, th. 3].

(II) There is a family $\Lambda = \{\lambda_{b,t}, b \in B, t \in T\}$ of measures on $S \times U$ with the properties:

(II_1) $\nu \otimes \lambda = \int_{B \times T} \lambda_{b,t} d(\kappa \otimes \mu)(b, t)$ and

(II_2) $\operatorname{Supp} \lambda_{b,t} \subset q^{-1}\{(b, t)\}$, $(b, t) \in B \times T$.

In fact, since both (B, κ), (T, μ) have SLP, there will exist [5]

a strict w-disintegration $(\lambda_b)_{b \in B}$ of ν and
a strict p-disintegration $(\lambda_t)_{t \in T}$ of λ.

If we put $\lambda_{b,t} := \lambda_b \otimes \lambda_t$, $b \in B$, $t \in T$, we easily verify that Λ is the desired family.

(III) In addition, Λ is a q-disintegration of $\nu \otimes \lambda$. Indeed, this fact is an immediate consequence of (I), (II) and the completion regularity of $\nu \otimes \lambda$ (cf. [7], prop. 1.5). However, from Lemma 2.8, the space $(S \times U, \nu \otimes \lambda)$ has SLP. It remains to show that there is a strong lifting for $(B \times T, \lambda \otimes \mu)$. This derives from the key projection theorem [8, th. 2.7].

2.11. REMARK. The example cited in 2.10 is a special case of a broader class of measure spaces as determined by A. and C. Ionescu Tulcea [5]. In fact, let $\{X_i, i \in J\}$ be a family of compact, metrizable spaces and μ a positive measure on $\Pi_{i \in J} X_i$ with $\operatorname{supp} \mu = \Pi_{i \in J} X_i$. Suppose that μ satisfies the general condition 5.2 of [5, th. 5, p. 118]. Then, taking a space (S, ν) with SLP and applying unchanged the proof of Lemma 2.8, we can verify that $(\Pi_{i \in J} X_i, \mu)$ has $\otimes$ SLP.

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